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ORIGINAL PAPER

Dynamical analysis of a chemostat model with delayed response in growth and pulse input in polluted environment

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Abstract In this paper, a chemostat model with delayed response in growth and pulse input in polluted environment is considered. Using the discrete dynamical system determined by the stroboscopic map, we obtain a microorganism-extinction periodic solution. Further, it is globally attractive. The permanent condition of the investigated system is also obtained by the theory on impulsive delay differential equation. Our results reveal that the delayed response in growth plays an important role on the outcome of the chemostat.

Keywords Chemostat model · Delayed response in growth · Pulse input in polluted environment · Extinction · Permanence

1 Introduction

The chemostat is a basic piece of laboratory apparatus. The advantages that certain of the biological parameters assumed to influence the outcomes can be controlled by the experimenters. The chemostat plays an important role in bioprocessing, such as ecology, microbiology, chemical engineering, etc. Smith and Waltman had made

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discussion about the chemostat model in [1]. The models contain discrete time delays which account for time which laps between uptaked of nutrient and the assimilation of nutrient into viable biomass. Smith and Waltman [2] and Kuang [3] discussed various aspects of models with discrete time delay. Freedman et al. [4] were the first to incorporate time delay in chemostat models. Ellermeyer [5] and Hsu et al. [6,7] analyzed a discrete time delay model with two competitive microorganisms for a single nutrient in a chemostat. Ellermeyer et al. [8] did a theoretical and empirical investigation of delayed growth response in the continuous culture of bacteria. Delays occur naturally in biological system by two obvious sources of delays: delays due to the cell cycle; and delays due to the possibility the organism stores the nutrient. Delays appear in a chemostat model in Bush and Cook [9]. They have investigated a model of growth of one organism in the chemostat with a delay in the intrinsic growth rate of the microorganism but with no delay in the substrate equation.

While the most threatening problem to the society is the change in both terrestrial and aquatic environment caused by the different kinds of stresses (temperature, toxicants/poflutants, etc.) affecting the long term survival of species, human life style and biodiversity of the habitat [10–12]. Presence of toxicant in the environments decreases the growth rate of species and its carrying capacity. In recent years, some investigations have been carried out to study the effect of toxicant on a single species population [13–16], and a lot of scholars have adopted mathematical modeling approach to study the influence of environmental pollution on the surviving of biological population [17,18]. Most of the previous work assumed that input of toxicant was continuous. The toxicants, however, are often emitted to the environment with regular pulse [19]. A lot of data have indicated that the use of agriculture chemicals may cause potential harm to the health of both human beings and living beings. If the spraying of agriculture chemicals can be regarded as time pulse discharge, the continuous input of toxin can be regarded discharged and replaced by an impulsive perturbations. In this case, though the discharge of toxin is transient, the influence of the toxin will last long. Therefore, it is very important that how controls the pulse input cycle of toxin to protect the population persistent existence. The system approximates conditions for plankton growth in lakes are in a chemostat form, where the limiting nutrients such as silica and phosphate are supplied from streams draining the watershed.

In recent years, the microbial continuous culture has been investigated in [7,8,20–24] and some interesting results were obtained. Many researchers indicated that it was important to consider models with periodic perturbations, since these models may be quite naturally exposed in many real world phenomena, for instance, food supply, mating habits. In fact, the perturbations such as floods and the drainage of sewage which are not suitable to be considered with continuity. These perturbations bring sudden changes to the system. Systems with sudden perturbations are involving an impulsive differential equations which have been studied intensively and systematically in [25,26]. While there are few papers [27,28] research the chemostat model with impulsive perturbations.

The organizations of the paper are as following. In Sect. 2, we introduce a chemostat model with delayed response in growth and pulse input in polluted environment. In Sect. 3, we present some preliminary results about the investigated model. Our main results are stated and proven in Sect. 4. Finally, we conclude with a brief discussion in Sect. 5.

2 The model

We investigate the following chemostat model with delayed response in growth and pulse input in polluted environment.

$$\begin{cases} \left.\begin{aligned} \frac{ds(t)}{dt} &= D(s^0 - s(t)) - P(s(t))x(t), \\ \frac{dx(t)}{dt} &= -Dx(t) + e^{-D\tau_1}P(s(t-\tau_1))x(t-\tau_1) - \beta c_0(t)x(t), \\ \frac{dc_0(t)}{dt} &= fc_e(t) - (g+m)c_0(t), \\ \frac{dc_e(t)}{dt} &= -hc_e(t), \end{aligned}\right\} t \neq n\tau, \\ \left.\begin{aligned} \triangle s(t) &= p_1, \\ \triangle x(t) &= 0, \\ \triangle c_0(t) &= 0, \\ \triangle c_e(t) &= p_2, \end{aligned}\right\} t = n\tau, n = 1, 2 \ldots, \\ (\varphi_1(\zeta), \varphi_2(\zeta), \varphi_3(\zeta), \varphi_4(\zeta)) \in C_+ = C([-\tau_1, 0], R_+^4), \varphi_i(0) > 0, i = 1, 2, 3, 4, \end{cases} \tag{2.1}$$

where $s(t)$ denotes the concentration of the substrate at time t. s^0 denotes the concentration of substrate in the feed bottle. $x(t)$ denotes the concentration of the microorganism at time t. $c_0(t)$ represents the concentration of toxicant in the microorganism at time t, $c_e(t)$ represents the concentration of toxicant in the environment at time t. τ_1 stands for the time delay in conversion of nutrient to biomass for the microorganism. As discussed in [5,24], $e^{-D\tau_1}x(t-\tau_1)$ represents the biomass of those microorganisms that consume nutrient τ_1 units of time prior to time t and that survive in the chemostat the τ_1 units of time necessary to complete the nutrient conversion process. D denotes the input rate from the feed bottle containing the substrate and the wash-out rate of substrate and microorganisms by products from the growth chamber. $P(s(t))$ indicates the consumption rate of nutrient by the microorganism. It is assumed that $P(0) = 0$, $P'(0) > 0$ and $P''(0) \leq 0$. β is the depletion rate coefficient of the microorganism population due to organismal pollutant concentration. $fc_e(t)$ is the organism's net uptake of toxicant from the environment at time t. $-gc_0(t)$ and $-mc_0(t)$ represents the elimination and depuration rates of toxicant in the organism at time t,respectively. $-hc_e(t)$ represents the totality of losses from the system environment including processes such as biological transformation, chemical hydrolysis, volatilization, microbial degradation and photosynthetic degradation at time t. τ is the period of the throwing in substrate concentration. $\triangle s(t) = s(t^+) - s(t)$ and $\triangle c_e(t) = c_e(t^+) - c_e(t)$, $p_1 \geq 0$ is the amount of the substrate concentration pulse at $t = n\tau, n \in Z_+$ and $Z_+ = \{1, 2, \ldots\}$, and $p_2 \geq 0$ is the amount of pulse input of toxicant concentration at $t = n\tau, n \in Z_+$ and $Z_+ = \{1, 2, \ldots\}$. The purpose of this paper is to prove that the system (2.1) has a microorganism-extinction periodic solution, further, it is globally attractive, and system (2.1) is permanence.

3 The lemmas

The solution of (2.1), denoted by $X(t) = (s(t), x(t), c_0(t), c_e(t))^T$, is a piecewise continuous function X: $R_+ \to R_+^4$, $X(t)$ is continuous on $(n\tau, (n+1)\tau]$, $n \in Z_+$ and $X(n\tau^+) = \lim_{t\to n\tau^+} X(t)$ exists. Obviously the global existence and uniqueness of solutions of (2.1) is guaranteed by the smoothness properties of f, which denotes the mapping defined by right-side of system (2.1) [25,26].

Before we have the main results. we need to give some lemmas which will be used in the next.

Lemma 3.1 [19] *Consider the following subsystem of* (2.1)

$$\begin{cases} \left.\begin{aligned} \frac{dc_0(t)}{dt} &= fc_e(t) - (g+m)c_0(t), \\ \frac{dc_e(t)}{dt} &= -hc_e(t), \end{aligned}\right\} t \neq n\tau, n \in Z^+, \\ \left.\begin{aligned} \triangle c_0(t) &= 0, \\ \triangle c_e(t) &= p_2, \end{aligned}\right\} t = n\tau, n = 1, 2 \ldots, n \in Z^+, \end{cases} \tag{3.1}$$

then, system (3.1) *has a unique positive τ-periodic solution $(\widetilde{c_0(t)}, \widetilde{c_e(t)})$, which is globally asymptotically stable, where*

$$\begin{cases} \widetilde{c_0(t)} = \widetilde{c_0(0)}e^{-(g+m)(t-n\tau)} + \dfrac{p_2 f(e^{-(g+m)(t-n\tau)} - e^{-h(t-n\tau)})}{(h-g-m)(1-e^{-h\tau})}, \\ \widetilde{c_e(t)} = \dfrac{p_2 e^{-h(t-n\tau)}}{1-e^{-h\tau}}, \\ \widetilde{c_0(0)} = \dfrac{p_2 f(e^{-(g+m)\tau} - e^{-h\tau})}{(h-g-m)(1-e^{-(g+m)\tau})(1-e^{-h\tau})}, \\ \widetilde{c_e(0)} = \dfrac{p_2}{1-e^{-h\tau}}, \end{cases} \tag{3.2}$$

Remark 3.1 From Lemma 3.1, we can obtain that $m_0 \leq c_0(t) \leq M_0$ and $m_e \leq c_e(t) \leq M_e$ for t large enough, where $m_0 = \frac{p_2 f(e^{-(g+m)\tau} - e^{-h\tau})}{(h-g-m)(e^{(g+m)\tau}-1)(1-e^{-h\tau})} - \varepsilon > 0$, $M_0 = \frac{p_2 f(e^{-(g+m)\tau} - e^{-h\tau})}{(h-g-m)(1-e^{-(g+m)\tau})(1-e^{-h\tau})} + \frac{p_2 f}{|h-g-m|(1-e^{-h\tau})} + \varepsilon$, $m_e = \frac{p_2 e^{-h\tau}}{1-e^{-h\tau}} - \varepsilon > 0$ and $M_e = \frac{p_2}{1-e^{-h\tau}} + \varepsilon$ for sufficiently small $\varepsilon > 0$.

Lemma 3.2 *Let $(\varphi_1(t), \varphi_2(t), \varphi_3(\zeta), \varphi_4(\zeta)) > 0$ for $-\tau_1 < t < 0$, then any solution of system* (2.1) *is strictly positive.*

Proof By uniqueness of solutions of system (2.1) and $s'(t) > 0$ whenever $s(t) = 0$,$t \neq n\tau$, and $s(n\tau^+) = s(n\tau) + p_1$, for $p_1 \geq 0$. It is easy to see that $s(t) > 0$ for all $t > 0$.

Secondly, we show that $x(t) > 0$ for all $t > 0$. Notice $x(t) > 0$, hence if there exists t_0 such that $x(t_0) = 0$, then $t_0 > 0$. Assume that t_0 is the first such time that $x(t) = 0$, that is $t_0 = inf\{t > 0 : x(t) = 0\}$, then $x'(t_0) = e^{-D\tau_1} P(s(t_0 - \tau_1))x(t_0 - \tau_1) > 0$. Hence for sufficiently small $\varepsilon > 0$, $x'(t - \varepsilon) > 0$. But by the definition of t_0, $x'(t_0 - \varepsilon) \leq 0$. This contradiction shows that $x(t) > 0$ for all $t > 0$.

Finally, from Remark 3.1, it is easy to prove that $c_0(t)$ and $c_e(t)$ are positive. □

Lemma 3.3 *There exists a positive constant* $s^0 + \frac{p_1 \exp(D\tau)}{\exp(D\tau)-1}$ *such that* $s(t) \leq s^0 + \frac{p_1 \exp(D\tau)}{\exp(D\tau)-1}$, $x(t) \leq s^0 + \frac{\exp(D\tau)}{\exp(D\tau)-1}$ *for each solution* $(s(t), x(t), c_0(t), c_e(t))$ *of* (2.1) *with all* t *large enough.*

Proof Define $V(t) = e^{-D\tau_1}s(t) + x(t+\tau_1)$, then $t \neq n\tau$, we have

$$D^+V(t) + DV(t) = Ds^0$$

when $t = n\tau$, $V(n\tau^+) = V(n\tau) + p_1 e^{-D\tau_1} \leq V(n\tau) + p_1$. By Lemma 2.2. (which can be seen in [11], Page 23), for $t \in (n\tau, (n+1)\tau]$ we have

$$V(t) \leq V(0)\exp(-Dt) + s^0(1-\exp(-Dt)) + \frac{p_1 \exp(-D(t-\tau))}{1-\exp(D\tau)} + \frac{p_1 \exp(D\tau)}{\exp(D\tau)-1}$$

$$\rightarrow s^0 + \frac{p_1 \exp(D\tau)}{\exp(D\tau)-1}, t \rightarrow \infty.$$

So $V(t)$ is uniformly ultimately bounded. Hence, by the definition of $V(t)$, there exists a constant $s^0 + \frac{p\exp(D\tau)}{\exp(D\tau)-1} > 0$ such that $s(t) \leq \left[s^0 + \frac{p\exp(D\tau)}{\exp(D\tau)-1}\right]e^{D\tau_1}$, $x(t) \leq s^0 + \frac{p\exp(D\tau)}{\exp(D\tau)-1}$ for t large enough. □

Lemma 3.4 [3] *Considering the following delay equation*

$$y'(t) = a_1 y(t-\tau) - a_2 y(t), \tag{3.3}$$

where $a_1, a_2, \tau > 0$; $y(t) > 0$ *for* $-\tau \leq t \leq 0$. *If* $a_1 < a_2$, $\lim_{t\rightarrow\infty} y(t) = 0$.

Lemma 3.5 [29] *Considering the following impulsive system*

$$\begin{cases} v'(t) = D(s^0 - v(t)), t \neq n\tau, \\ v(n\tau^+) = v(n\tau) + p_1, t = n\tau, n = 1, 2 \ldots. \end{cases} \tag{3.4}$$

where $a > 0, b > 0$. Then system (2.4) has a unique positive periodic solution $\widetilde{v(t)} = s^0 + \frac{p_1 e^{-D(t-n\tau)}}{1-e^{-D\tau}}$, $t \in (n\tau, (n+1)\tau]$, $n \in Z_+$, which is globally asymptotically stable.

4 Dynamical behaviors of system (2.1)

According to Lemma 3.1 and Lemma 3.5, we know that (2.1) has a microorganism-extinction periodic solution $(\widetilde{s(t)}, 0, \widetilde{c_0(t)}, \widetilde{c_e(t)})$. In this section, we will obtain the sufficient condition of the global attractivity of microorganism-extinction periodic solution $(\widetilde{s(t)}, 0, \widetilde{c_0(t)}, \widetilde{c_e(t)})$ of system (2.1).

Theorem 4.1 *If*

$$p_1 \leq (e^{-D\tau}-1)\left\{P^{-1}\left[e^{-D\tau_1}\left(D+\frac{\beta p_2 f(e^{-(g+m)\tau}-e^{-h\tau})}{(h-g-m)(e^{(g+m)\tau}-1)(1-e^{-h\tau})}\right)\right]-s^0\right\},$$

holds, the microorganism-extinction periodic solution $(\widetilde{s(t)}, 0, \widetilde{c_0(t)}, \widetilde{c_e(t)})$ *of* (2.1) *is globally attractive.*

Proof Since $p_1 \leq (e^{-D\tau}-1)\left\{P^{-1}\left[e^{-D\tau_1}\left(D+\frac{\beta p_2 f(e^{-(g+m)\tau}-e^{-h\tau})}{(h-g-m)(e^{(g+m)\tau}-1)(1-e^{-h\tau})}\right)\right]-s^0\right\}$, we can choose ε_0 sufficiently small such that

$$e^{-D\tau_1} P\left(s^0 + \frac{p_1}{e^{D\tau}-1} + \varepsilon_0\right) < D + \beta\left(\widetilde{c_0(t)} - \varepsilon_0\right).$$

It follows from that the first equation of system (2.1) that $\frac{ds(t)}{dt} \leq D(s^0 - s(t))$. So we consider the following comparison impulsive differential system

$$\begin{cases} \dfrac{dx_1(t)}{dt} = D(s^0 - x_1(t)), t \neq n\tau, \\ \triangle x_1(t) = p_1, t = n\tau, \\ x_1(0^+) = s(0^+), \end{cases} \tag{4.1}$$

In view of Lemma 3.5, we obtain that the periodic solution of system (4.1)

$$\widetilde{x_1(t)} = s^0 + \frac{p_1 e^{-D(t-n\tau)}}{1-e^{-D\tau}}, t \in (n\tau, (n+1)\tau], n \in Z_+, \tag{4.2}$$

which is globally asymptotically stable.

From Lemma 2.4. and comparison theorem of impulsive equation [16], we have $s(t) \leq x_1(t)$ and $x_1(t) \to \widetilde{x_1(t)}$ as $t \to \infty$. Then there exists an integer $k_2' > k_1$, $t > k_2'$ such that

$$s(t) \leq x_1(t) \leq \widetilde{x_1(t)} + \varepsilon_0, n\tau < t \leq (n+1)\tau, n > k_2',$$

that is

$$s(t) < \widetilde{x_1(t)} + \varepsilon_0 \leq s^0 + \frac{p_1}{e^{D\tau}-1} + \varepsilon_0 \stackrel{\triangle}{=} \varrho, n\tau < t \leq (n+1)\tau, n > k_2'.$$

Because $s(t)$ and $x(t)$ cannot affect the subsystem (3.1) of system (2.1), and from Lemma 3.1, we obtain that $c_0(t) \geq \widetilde{c_0(t)} - \varepsilon_0'$ for $n\tau < t \leq (n+1)\tau, n > k_2''$, that is, $c_0(t) \geq m_0$ for $n\tau < t \leq (n+1)\tau, n > k_2''$. For convenience, assuming $\varepsilon_0' = \varepsilon_0$, $k_2 = \max\{k_2', k_2''\}$, we have $s(t) < \varrho, c_0(t) \geq m_0, n\tau < t \leq (n+1)\tau, n > k_2, k_2 > k_1$.

From the second equation of system (2.1), we get

$$\frac{dx(t)}{dt} \leq e^{-D\tau_1} P(\varrho)x(t-\tau_1) - (D+\beta m_0)x(t), t > n\tau + \tau_1, n > k_2, \tag{4.3}$$

Considering the following comparison differential system

$$\frac{dy(t)}{dt} = e^{-D\tau_1} P(\varrho) y(t-\tau_1) - (D + \beta m_0) y(t), t > n\tau + \tau_1, n > k_2, \quad (4.4)$$

we have $e^{-D\tau_1} P(\varrho) < D + \beta m_0$. According to Lemma 3.4, we have $\lim_{t\to\infty} y(t) = 0$.

Let $(s(t), x(t), c_0(t), c_e(t))$ be the solution of system (2.1) with initial conditions and $x(\zeta) = \varphi_2(\zeta)(\zeta \in [-\tau_1, 0])$, $y(t)$ is the solution of system (4.4) with initial conditions $y(\zeta) = \varphi_3(\zeta)(\zeta \in [-\tau_1, 0])$. By the comparison theorem, we have

$$\lim_{t\to\infty} x(t) < \lim_{t\to\infty} y(t) = 0,$$

Incorporating into the positivity of $x(t)$, we know that $\lim_{t\to\infty} x(t) = 0$, Therefore, for any $\varepsilon_1 > 0$ (sufficiently small), there exists an integer $k_3 (k_3\tau > k_2\tau + \tau_1)$ such that $x(t) < \varepsilon_1$ for all $t > k_3\tau$.

For system (2.1), we have

$$D[s^0 - (1 + \varepsilon_1 P'(0)) s(t)] \le \frac{ds(t)}{dt} \le D(s^0 - s(t)), \quad (4.5)$$

Then we have $z_1(t) \le s(t) \le z_2(t)$ and $z_1(t) \to \widetilde{x_1(t)}, z_2(t) \to \widetilde{x_1(t)}$ as $t \to \infty$. while $z_1(t)$ and $z_2(t)$ are the solutions of

$$\begin{cases} \dfrac{dz_1(t)}{dt} = D[s^0 - (1 + \varepsilon_1 P'(0)) z_1(t)], t \ne n\tau, \\ z_1(t^+) = z_1(t) + p_1, t = n\tau, \\ z_1(0^+) = s(0^+), \end{cases} \quad (4.6)$$

and

$$\begin{cases} \dfrac{dz_2(t)}{dt} = D(s^0 - z_2(t)), t \ne n\tau, \\ z_2(t^+) = z_2(t) + p_1, t = n\tau, \\ z_2(0^+) = s(0^+), \end{cases} \quad (4.7)$$

respectively. For $n\tau < t \le (n+1)\tau$, $\widetilde{z_1(t)} = \frac{1}{1+\varepsilon_1 P'(0)} \left[s^0 + \frac{p_1 e^{-D(1+\varepsilon P'(0))(t-\tau)}}{1-e^{-D(1+\varepsilon_1 P'(0))\tau}} \right]$. Therefore, for any $\varepsilon_2 > 0$. there exists a integer $k_4, n > k_4$ such that $\widetilde{z_1(t)} + \varepsilon_2 < s(t) < \widetilde{s(t)} - \varepsilon_2$, Let $\varepsilon_1 \to 0$, so we have $\widetilde{s(t)} - \varepsilon_2 < s(t) < \widetilde{s(t)} + \varepsilon_2$, for t large enough, which implies $s(t) \to \widetilde{s(t)}$ as $t \to \infty$.

Because $s(t)$ and $x(t)$ cannot affect the subsystem (3.1) of system (2.1), and from Lemma 3.1, we obtain that $c_0(t) \to \widetilde{c_0(t)}$ and $c_e(t) \to \widetilde{c_e(t)}$ as $t \to \infty$. □

Definition 4.2 system (2.1) is said to be permanent, if there are constants $m, M, m_0, M_0, m_e, M_e > 0$ (independent of initial value) and a finite time T_0 such that for all solutions $(s(t), x(t), c_0(t), c_e(t))$ with all initial values $s(t) > 0, x(0^+) > 0$,

$m \leq s(t) < M, m \leq x(t) \leq M, m_0 \leq x(t) \leq M_0, m_e \leq x(t) \leq M_e$ holds for all $t \geq T_0$. Here T_0 may depend on the initial values $(s(0^+), x(0^+), c_0(0^+), c_e(0^+))$.

Theorem 4.3 *If*

$$p_1 \geq (e^{D(1+x^*P'(0))\tau} - 1) \times \left[P^{-1}\left[\left(D+\frac{\beta p_2 f(e^{-(g+m)\tau}-e^{-h\tau})}{(h-g-m)(1-e^{-(g+m)\tau})(1-e^{-h\tau})}+\frac{p_2 f}{|h-g-m|(1-e^{-h\tau})}\right)e^{D\tau_1}\right] \times (1+x^*P^*(0)) - s^0\right],$$

holds, there is a positive constant q such that each positive solution $(s(t), x(t), c_0(t), c_e(t))$ of (2.1) *satisfies $x(t) \geq q$, for t large enough. Where x^* is determined by the equation*

$$s^0(e^{D(1+x^*P'(0))\tau} - 1) + p_1 = [e^{D(1+x^*P'(0))\tau} - 1] \times P^{-1} \times \left[\left(D+\frac{\beta p_2 f(e^{-(g+m)\tau}-e^{-h\tau})}{(h-g-m)(1-e^{-(g+m)\tau})(1-e^{-h\tau})}+\frac{p_2 f}{|h-g-m|(1-e^{-h\tau})}\right)e^{D\tau_1}\right] \times [1+x^*P'(0)].$$

Proof The second equation of system (2.1) can be rewritten as

$$\frac{dx(t)}{dt} = [e^{-D\tau}P(s(t)) - (D+\beta c_0(t))]x(t) - e^{-D\tau_1}\frac{d}{dt}\int_{t-\tau_1}^{t} P(s(u))x(u)du, \tag{4.8}$$

Let us consider any positive solution $(s(t), x(t), c_0(t), c_e(t))$ of system (2.1). According to (4.8), $V(t)$ is defined as

$$V(t) = x(t) + e^{-D\tau_1}\int_{t-\tau_1}^{t} P(s(u))x(u)du,$$

We calculate the derivative of $V(t)$ along the solution of (2.1)

$$\frac{dV(t)}{dt} = [e^{-D\tau}P(s(t)) - (D+\beta c_0(t))]x(t), \tag{4.9}$$

Since $p \geq (e^{D(1+x^*P'(0))\tau} - 1)[P'(De^{D\tau_1})(1+x^*P^*(0)) - s^0]$, we can easily know that there exists sufficiently small $\varepsilon > 0$ such that

$$e^{-D\tau}P\left(\frac{1}{1+x^*P'(0)}\left[s^0+\frac{p_1}{e^{D(1+x^*P'(0))\tau}-1}\right]+\varepsilon\right)$$

$$> D+\beta\left(\frac{p_2 f(e^{-(g+m)\tau}-e^{-h\tau})}{(h-g-m)(1-e^{-(g+m)\tau})(1-e^{-h\tau})}+\frac{p_2 f}{|h-g-m|(1-e^{-h\tau})}+\varepsilon\right).$$

We claim that for any $t_0 > 0$, it is impossible that $x(t) < x^*$ for all $t > t_0$. Suppose that the claim is not valid. Then there is a $t_0 > 0$ such that $x(t) < x^*$ for all $t > t_0$. It follows from the first equation of (2.1) that for all $t > t_0$

$$\frac{ds(t)}{dt} > D[s^0 - (1 + P'(0)x^*)s(t)], \tag{4.10}$$

Consider the following comparison impulsive system for all $t > t_0$

$$\begin{cases} \frac{dv(t)}{dt} = D[s^0 - (1 + P'(0)x^*)v(t)], t \neq n\tau, \\ \triangle v(t) = p_1, t = n\tau, \end{cases} \tag{4.11}$$

By Lemma 3.5, we obtain $\widetilde{v(t)} = \frac{1}{1+x^*P'(0)}\left[s^0 + \frac{p_1 e^{-D(1+x^*P'(0))(t-n\tau)}}{1-e^{-D(1+x^*P'(0))\tau}}\right], n\tau < t \leq (n+1)\tau$ is the unique positive periodic solution of (4.11) which is globally asymptotically stable. By the comparison theorem for impulsive differential equation [11], we know that there exists $t_1(> t_0 + \tau_1)$ such that the inequality $s(t) \geq \widetilde{v(t)} + \varepsilon$ holds for $t \geq t_1$, thus $s(t) \geq \frac{1}{1+x^*P'(0)}\left[s^0 + \frac{p_1}{e^{D(1+x^*P'(0))\tau}-1}\right] + \varepsilon$ for all $t \geq t_1$. For convenience, we make notation as $\sigma \triangleq \frac{1}{1+x^*P'(0)}\left[s^0 + \frac{p}{e^{D(1+x^*P'(0))\tau}-1}\right] + \varepsilon$. So we have

$$e^{-D\tau_1}P(\sigma) > D + \beta M_0,$$

then we have

$$V'(t) > x(t)[e^{-D\tau_1}P(\sigma) - (D + \beta M_0)],$$

for all $t > t_1$. Set $x^m = \min_{t\in[t_1,t_1+\tau_1]} x(t)$, we will show that $x(t) \geq x^m$ for all $t \geq t_1$. Suppose the contrary, then there is a $T_0 > 0$ such that $x(t) \geq x^m$ for $t_1 \leq t \leq t_1 + \tau_1 + T_0$, $x(t_1 + \tau_1 + T_0) = x^m$ and $x'(t_1 + \tau_1 + T_0) < 0$. Hence, the first equation of system (1.1) imply that

$$\begin{aligned} x'(t_1 + \tau_1 + T_0) &= e^{-D\tau_1}P(s(t_1 + \tau_1 + T_0)x(t_1 + \tau_1 + T_0) \\ &\quad -[D + \beta c_0(t_1 + \tau_1 + T_0)]x(t_1 + \tau_1 + T_0), \\ &\geq [e^{-D\tau_1}P(\sigma) - (D + \beta M_0)]x^m > 0. \end{aligned}$$

This is a contradiction. Thus, $x(t) \geq x^m$ for all $t > t_1$. As a consequence, Then $V'(t) > x^m[e^{-D\tau_1}P(\sigma) - (D + \beta M_0)] > 0$ for all $t > t_1$. This implies that as $t \to \infty$, $V(t) \to \infty$. It is a contradiction to $V(t) \leq M\left(1 + \tau_1 e^{-D\tau_1}P\left(s^0 + \frac{p\exp(D\tau)}{\exp(D\tau)-1}\right)\right)$. Hence, the claim is complete.

By the claim, we are left to consider two case. First, $x(t) \geq x^*$ for all t large enough. Second, $x(t)$ oscillates about x^* for t large enough.

Define

$$q = \min\left\{\frac{x^*}{2}, q_1\right\}, \tag{4.12}$$

where $q_1 = x^* e^{-(D+\beta M_0)\tau_1}$. We hope to show that $x(t) \geq q$ for all t large enough. The conclusion is evident in first case. For the second case, let $t^* > 0$ and $\xi > 0$ satisfy $x(t^*) = x(t^* + \xi) = x^*$ and $x(t) < x^*$ for all $t^* < t < t^* + \xi$ where t^* is sufficiently large such that $x(t) > \sigma$ for $t^* < t < t^* + \xi$, $x(t)$ is uniformly continuous. The positive solutions of (1.1) are ultimately bounded and $x(t)$ is not affected by impulses. Hence, there is a $T(0 < t < \tau_1$ and T is dependent of the choice of $t^*)$ such that $x(t^*) > \frac{x^*}{2}$ for $t^* < t < t^* + T$. If $\xi < T$, there is nothing to prove. Let us consider the case $T < \xi < \tau_1$. Since $x'(t) > -(D+\beta M_0)x(t)$ and $x(t^*) = x^*$, it is clear that $x(t) \geq q_1$ for $t \in [t^*, t^* + \tau_1]$. Then, proceeding exactly as the proof for the above claim. We see that $x(t) \geq q_1$ for $t \in [t^* + \tau_1, t^* + \xi]$. Because the kind of interval $t \in [t^*, t^* + \xi]$ is chosen in an arbitrary way (we only need t^* to be large). We concluded $x(t) \geq q$ for all large t. In the second case. In view of our above discussion, the choice of q is independent of the positive solution, and we proved that any positive solution of (2.1) satisfies $x(t) \geq q$ for all sufficiently large t. □

Theorem 4.4 *If*

$$p_1 \geq (e^{D(1+x^*P'(0))\tau} - 1) \times \left[P^{-1} \left[\left(D + \frac{\beta p_2 f (e^{-(g+m)\tau} - e^{-h\tau})}{(h-g-m)(1-e^{-(g+m)\tau})(1-e^{-h\tau})} + \frac{p_2 f}{|h-g-m|(1-e^{-h\tau})} \right) e^{D\tau_1} \right] \times (1 + x^* P^*(0)) - s^0 \right], \tag{4.13}$$

holds, system (2.1) *is permanent.*

Proof Denote $(s(t), x(t), c_0(t), c_e(t))$ be any solution of system (2.1). From the first equation of system (2.1) and theorem 4.3, we have

$$\begin{cases} \dfrac{ds(t)}{dt} \geq D \left\{ s^0 - \left[1 + \left(s^0 + \dfrac{p_1 \exp(D\tau)}{\exp(D\tau) - 1} \right) P'(0) \right] s(t) \right\}, t \neq n\tau, \\ \triangle s(t) = p_1, n = \tau, \end{cases} \tag{4.14}$$

By the same argument as those in the proof of theorem 4.1, we have that $\lim_{t\to\infty} x_1(t) \geq w$, where $w = \frac{1}{1+(s^0+\frac{p_1 \exp(D\tau)}{\exp(D\tau)-1})P'(0)} \left[s^0 + \frac{p_1 e^{-D[1+(s^0+\frac{p_1 \exp(D\tau)}{\exp(D\tau)-1})P'(0)](t-\tau)}}{1-e^{-D(1+(s^0+\frac{p_1 \exp(D\tau)}{\exp(D\tau)-1})P'(0))\tau}} \right] - \varepsilon$.

By Remark 3.1 and the above discussion, system (2.1) is permanent. □

5 Discussion

A chemostat system with delayed response in growth and pulse input in polluted environment is investigated in this paper. We obtain that the microorganism-extinction periodic solution of system (2.1) is globally attractive. The permanent condition of the system (2.1) is also obtained. From Theorem 4.1 and Theorem 4.4, we can also guess that there must exist a threshold p_1^*. If $p_1 < p_1^*$, the microorganism-extinction periodic solution $(\widetilde{s(t)}, 0, \widetilde{c_0(t)}, \widetilde{c_e(t)})$ of (2.1) is globally attractive. If $p_1 > p_1^*$, system (2.1) is permanent, or from Theorem 4.1 and Theorem 4.4, we can easily guess that there must exist a threshold τ_1^*. If $\tau_1 < \tau_1^*$, the microorganism-extinction periodic solution $(\widetilde{s(t)}, 0, \widetilde{c_0(t)}, \widetilde{c_e(t)})$ of (2.1) is globally attractive. If $\tau_1 > \tau_1^*$, system (2.1) is permanent. The results show the delayed response in growth (which is depicted by τ_1) plays an important role for the permanence of system (2.1), and the pulse input on substrate affects the dynamical behaviors of system (2.1).

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